Michigan Autumn Take-Home (MATH) Challenge

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1. Assume that a convex polyhedron has n vertices, n faces and 150 edges. For the face f_i, $1 \leq i \leq n$, let $\alpha(f_i)$ denote the sum of the interior angles of f_i. Find $\sum_{i=1}^{n} \alpha(f_i)$.

2. *NCAA basketball pool.* There are 64 teams who play single elimination tournament, hence 6 rounds, and you have to predict all the winners in all 63 games. Your score is then computed as follows: 32 points for correctly predicting the final winner, 16 points for each correct finalist, and so on, down to 1 point for every correctly predicted winner for the first round. (The maximum number of points you can get is thus 192.) Knowing nothing about any team, you flip fair coins to decide every one of your 63 bets. Compute the expected number of points.

3. Find all positive integers n, such that n^2 is divisible by $n + 2019$ (Note: $2019 = 3 \times 673$, and 673 is a prime number.)

4. Prove that given 100 different positive integers such that none of them is a multiple of 100, it is always possible to choose several of them such that the last two digits of their sum are zeros.

5. *Chasing cats puzzle.* There are n cats sitting at the n different vertices of a regular polygon, with length of each side a. Each of those cats start chasing the other cat in the clockwise direction. The speed of the cats are same and constant and they continuously change their direction in a manner that they are always heading straight to the other cat.

   (a) How long will it take for the cats to catch each other at the center of the polygon?

   (b) Each cat moves along a curve starting from a vertex, and ending at the center of the polygon. Find the length of those curves.

6. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{n}, & \text{if } x = \frac{m}{n} \in \mathbb{Q},\ \gcd(m,n) = 1,\ n > 0 \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

(a) Prove that for all $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = 0$. (Thus f is continuous at every irrational number.)

(b) For $x_0 \notin \mathbb{Q}$, prove that f is NOT differentiable at x_0.

7. Let f be a real-valued function such that f, f', and f'' are all continuous on [0,1]. Prove that the series $\sum_{k=1}^{\infty} f(\frac{1}{k})$ is convergent, if and only if $f(0) = 0$, and $f'(0) = 0$.

8. Find a function $f(x)$ that is never 0 and satisfies the following integral equation for all x:

$$\left(\int_0^x f(t)\,dt\right)^2 = \int_0^x [f(t)]^2\,dt - 2\int_0^x [f(t)]\,dt.$$

9. A free finitely generated group Γ of rank 2 is the group of all words generated by the two letters γ_1 and γ_2. Each γ not equal to the identity element e, can be uniquely written as $\gamma_1^{p_1}\gamma_2^{p_2}\gamma_1^{p_3}\cdots\gamma_{i_k}^{p_k}$, or as $\gamma_2^{p_1}\gamma_1^{p_2}\gamma_2^{p_3}\cdots\gamma_{i_k}^{p_k}$, where $p_1, \cdots, p_k$ are non-zero integers, and $i_k = 1, 2$. The norm norm $\|\gamma\|$ is then defined to be $\sum_{i=1}^{k} |p_i|$. If Γ is Abelian the words can be simplified as $\gamma_1^p\gamma_2^q$, $p, q \in \mathbb{Z}$. *The ball of radius* r centered at e, $B(r)$, consists of all $\gamma \in \Gamma$ with $\|\gamma\| \le r$. We denote by $\#B(r)$ the number of elements in $B(r)$. Prove that:

(a) If Γ is Abelian, $\#B(r) = 2N^2 + 2N + 1$, where $N = \lfloor r \rfloor$.

(b) If Γ is non-Abelian, $\#B(r) = 2 \cdot 3^N - 1$, where $N = \lfloor r \rfloor$.

10. The n-dimensional unit sphere S^n, is the set of all points in $\mathbb{R}^{n+1}$ of distance 1 from the origin, that is $S^n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\}$. The intersection of every n-dimensional vector space $V \subset \mathbb{R}^{n+1}$ with S^n is a $(n-1)$-dimensional unit sphere, called a *great sphere* of S^n. Every great sphere divides S^n into two hemisphere. A hemisphere together with its boundary, is a *closed* hemisphere. Prove that given any $n + 3$ points in S^n, there is a closed hemisphere that contains $n + 2$ of them.

Michigan Autumn Take-Home (MATH) Challenge Solutions

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1. Assume that a convex polyhedron has n vertices, n faces and 150 edges. For the face f_i, $1 \leq i \leq n$, let $\alpha(f_i)$ denote the sum of the interior angles of f_i. Find $\sum_{i=1}^{n} \alpha(f_i)$.

   Solution: Let $e(f_i)$ be the number of edges for the face f_i. Then $\alpha(f_i) = (e(f_i) - 2)\pi$; thus it follows that

   $$\sum_{i=1}^{n} \alpha(f_i) = \sum_{i=1}^{n}(e(f_i) - 2)\pi = \pi \sum_{i=1}^{n} e(f_i) - 2n\pi.$$

   Since in $\sum_{i=1}^{n} e(f_i)$ every edge is counted twice, it follows that $\sum_{i=1}^{n} e(f_i) = 300$.

   On the other hand, by Euler's formula, $v - e + f = 2$, and noting $v = f = n$ we have $2n - 150 = 2$, so $n = 76$. Thus,

   $$\sum_{i=1}^{n} \alpha(f_i) = 300\pi - 152\pi = 148\pi.$$

2. *NCAA basketball pool.* There are 64 teams who play single elimination tournament, hence 6 rounds, and you have to predict all the winners in all 63 games. Your score is then computed as follows: 32 points for correctly predicting the final winner, 16 points for each correct finalist, and so on, down to 1 point for every correctly predicted winner for the first round. (The maximum number of points you can get is thus 192.) Knowing nothing about any team, you flip fair coins to decide every one of your 63 bets. Compute the expected number of points.

   Solution: If you have n rounds and 2^n teams, the answer is $\frac{1}{2}(2^n - 1)$, so $n = 6$ gives 31.5.

   Comment: This is an example of how useful linearity of expectation is. Fix a game g and let I_g be the indicator of the event that you collect points in this game, that is, that you correctly predict this winner. If $s = s(g)$ is the game's round, then your winnings on this game are $2^{s-1} I_g$. However, $E(I_g)$ is the probability that you have correctly predicted the winner of this game in this *and* all previous rounds, that is, 2^{-s}. Your expected winnings on this game are then $2^{s-1} \cdot 2^{s} = \frac{1}{2}$. This is independent of g, so your answeer is half the total number of games.

3. Find all positive integers n, such that n^2 is divisible by $n+2019$. (Note: $2019 = 3\times 673$, and 673 is a prime number.)

Solution: Since $\gcd(n+2019, n) = \gcd(2019, n)$ and we must have $(n+2019)\,|\,n^2$, it follows that $\gcd(n+2019, n) > 1$. Note that $2019 = 3\times 673$, so we may consider 3 cases.

A. $\mathbf{gcd(n, 2019) = 3}$: Then $n = 3k$, where $\gcd(k, 2019) = 1)$, $k > 0$. It follows that $(k+673)\,|\,3k^2$, but $\gcd(k+673, k) = \gcd(673, k) = 1$, hence $(k+673)\,|\,3$, which is impossible.

B. $\mathbf{gcd(n, 2019) = 673}$: Then $n = 673k$, where $\gcd(k, 2019) = 1)$, $k > 0$. Thus we have $(k+3)\,|\,673k^2$. Since $\gcd(k+3, k) = \gcd(3, k) = 1$, we must have $(k+3)\,|\,673$, so $k+3 = 673$ and $k = 670$. Thus $n = 673\cdot 670 = 450,910$.

C. $\mathbf{gcd(n, 2019) = 2019}$: Then $n = 2019k$, where $k > 0$. Thus we have $(k+1)\,|\,2019k^2$. Since $\gcd(k+1, k^2) = 1$, it follows that $(k+1)\,|\,2019$, hence $k+1 = 3, 673$, or 2019. This produces the following numbers for n:

   - $n = 2019\times 2 = 4038$.
   - $n = 2019\times 672 = 1,356,768$.
   - $n = 2019\times 2018 = 4,074,342$.

So there are 4 answers.

4. Prove that given 100 different positive integers such that none of them is a multiple of 100, it is always possible to choose several of them such that the last two digits of their sum are zeros.

Solution: Label the numbers $\{a_1, a_2, \ldots, a_{100}\}$; by assumption, $a_i \not\equiv 0 \mod 100$. Let $S_i = \sum_{j=1}^{i} a_j$ and define r_j to be the remainder of S_j mod 100, so $S_i \equiv r_i \mod 100$, where $0 \le r_i \le 99$.

If there is an index i such that $r_i = 0$, then S_i has its last two digits 0 and we are done. If for all $i, i = 1, \ldots, 100$, we have $0 < r_i \le 99$, then by the pigeonhole principle there are indices i_1, j_1 such that $r_{i_1} = r_{j_1}$. Without loss of generality, we may assume $i_1 < j_1$. Thus it follows that $S_{j_1} - S_{i_1} \equiv 0 \mod 100$. Hence

$$\sum_{j=i_1+1}^{j_1} a_j \equiv 0 \mod 100,$$

completing the proof.

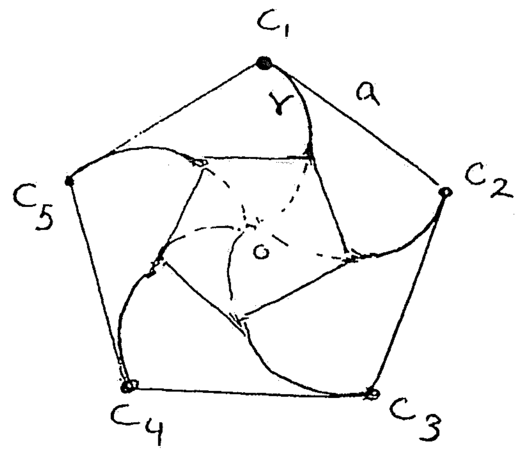
5. *Chasing cats puzzle*. There are n cats sitting at the n different vertices of a regular polygon, with length of each side a. Each of those cats start chasing the other cat in the clockwise direction. The speed of the cats are same and constant and they continuously change their direction in a manner that they are always heading straight to the other cat.

   (a) How long will it take for the cats to catch each other at the center of the polygon?

(b) Each cat moves along a curve starting from a vertex, and ending at the center of the polygon. Find the length of those curves.

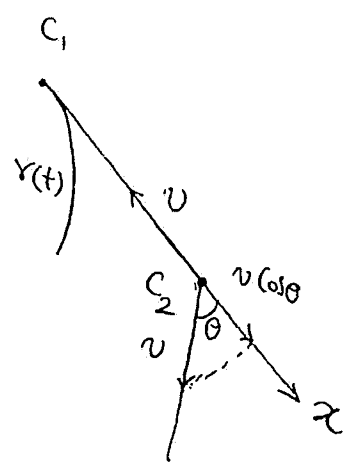
Solution:

(a) Assume the speed of each cat is v. A picture of the chasing cats for $n = 5$ is shown here.



We only need to solve the problem for one of the cats, say c_1. Note that, by assumption, the cats always stay at the vertices of the regular polygon that is continuously shrinking.

Define a moving coordinate system with its origin at the position of cat c_1 so that the x-axis is tangent to the trajectory of the cat, as in this picture.



The relative velocity of cat c_2 would be $v - v\cos\theta = v(1 - \cos\theta)$, where $\theta = \dfrac{2\pi}{n}$ is the exterior angle of the polygon.

Thus the time it takes for the cats c_1, c_2 to meet at the center of the polygon is

$$T = \frac{a}{v \cdot \left(\left(1 - \cos\left(\frac{2\pi}{n}\right)\right)\right)}.$$

(b) If $\gamma(t)$ is the trajectory of cat c_1, $|\gamma'(t)| = v$ for all t, thus

$$\ell(\gamma) = \int_0^T |\gamma'(t)|\, dt = T \cdot v = \frac{a}{1 - \cos\left(\frac{2\pi}{n}\right)}.$$

6. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in \mathbb{Q},\ \gcd(m,n) = 1,\ n > 0 \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

(a) Prove that for all $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = 0$. (Thus f is continuous at every irrational number.)

(b) For $x_0 \notin \mathbb{Q}$, prove that f is NOT differentiable at x_0.

Solution:

(a) We prove that for any real number x_0, $\lim_{x \to x_0} f(x) = 0$.
For a fixed number n, let $I_n = \left[\frac{m-1}{n}, \frac{m}{n}\right]$ be an interval such that $\frac{m-1}{n} \le x_0 < \frac{m}{n}$. Assume that $\epsilon > 0$ is given. Choose N large enough so that $N > \frac{1}{\epsilon}$. If $x_0 \notin \mathbb{Q}$ then let

$$I = \bigcap_{n=1}^{N} I_n.$$

Every rational number in I then has a denominator larger than N. Thus $x \in I$ implies $|f(x)| < \frac{1}{N} < \epsilon$. If $x_0 = \frac{m_0}{n_0} \in \mathbb{Q}$, let $J = \left(\frac{m_0 - 1}{n}, \frac{m_0+1}{n}\right)$ and let

$$I = J \cap \left(\bigcap_{n=1, n \neq n_0}^{N} I_n \right).$$

Then every rational number in I, other than x_0, has a denominator larger than N. Thus $x \in I$ implies $|f(x)| < \frac{1}{N} < \epsilon$. Consequently, we have $\lim_{x \to x_0} f(x) = 0$. If $x_0 \in \mathbb{Q}$ is nonzero, then $f(x_0) \neq 0$,, so $\lim_{x \to x_0} f(x) \neq f(x_0)$ and $f(x)$ is not continuous at x_0.

(b) For a rational number $x = \frac{m}{n}$ close to x_0,

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\frac{1}{n} - 0}{\frac{m}{n} - x_0} = \frac{-1}{nx_0 - m}.$$

Letting $m = \lfloor nx_0 \rfloor$, we have that $nx_0 - m$ is the fractional part of nx_0, and it is well-known that for $x_0 \notin \mathbb{Q}$ the set $\{nx_0\}$ is dense in $(0,1)$. Thus, given any number $c \in (0,1)$, we can choose a sequence n_k such that the fractional part of $n_k x_0 \to c$. Now, if

$$x_k = \frac{\lfloor n_k x_0 \rfloor}{n_k},$$

then $x_k \to x_0$ and

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{x_k - x_0} = \lim_{k\to\infty} \frac{-1}{n_k x_0 - \lfloor n_k x_0 \rfloor} = \frac{-1}{c}.$$

Since c is arbitrary, we conclude that $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ does not exist.

7. Let f be a real-valued function such that f, f', and f'' are all continuous on [0,1]. Prove that the series $\sum_{k=1}^{\infty} f(\frac{1}{k})$ is convergent, if and only if $f(0) = 0$, and $f'(0) = 0$.

Solution: Let $f''(0) = a$ and define $f(x) = \frac{a}{2}x^2$ if $x < 0$. Then f, f', and f'' are continuous on $(-\infty, 1]$.

First, assume $f(0) = f'(0) = 0$. Using the identity

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2},$$

it follows that

$$\lim_{k\to\infty} \frac{f\left(0+\frac{1}{k}\right) - 2f(0) + f\left(0-\frac{1}{k}\right)}{\left(\frac{1}{k}\right)^2} = a,$$

or

$$\lim_{k\to\infty} \frac{f(1/k)}{\left(\frac{1}{k}\right)^2} = \left|\frac{a}{2}\right|.$$

Now since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, the comparison test shows that $\sum_{k=1}^{\infty} f\left(\frac{1}{k}\right)$ is absolutely convergent.

Conversely, assume $\sum_{k=1}^{\infty} f\left(\frac{1}{k}\right)$ is convergent. Then the nth-term test implies that

$$f(0) = \lim_{k\to\infty} f\left(\frac{1}{k}\right) = 0.$$

We have

$$f'(0) = \lim_{k\to\infty} \frac{f\left(\frac{1}{k}\right) - f(0)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{f\left(\frac{1}{k}\right)}{\frac{1}{k}}.$$

Suppose that $f'(0) = b \neq 0$. Then

$$\lim_{k\to\infty} \frac{f\left(\frac{1}{k}\right) - f(0)}{\frac{1}{k}} = b \neq 0,$$

in particular, for all large enough k, $f(1/k)$ are all positive (or negative), so convergence and absolute convergence are equivalent. Since $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent, the comparison test implies that $\sum_{k=1}^{\infty} \left|f\left(\frac{1}{k}\right)\right|$ is divergent, thus so is $\sum_{k=1}^{\infty} f\left(\frac{1}{k}\right)$. This is a contradiction.

8. Find a function $f(x)$ that is never 0 and satisfies the following integral equation for all x:

$$\left(\int_0^x f(t)\,dt\right)^2 = \int_0^x [f(t)]^2\,dt - 2\int_0^x [f(t)]\,dt\,.$$

Solution: Let

$$y(x) = \int_0^x f(t)\,dt;$$

by the Fundamental theorem of Calculus, $y'(x) = f(x)$. Differentiating the integral equation, it follows that $2yy' = (y')^2 - 2y'$. Since $y'(x) = f(x)$ is never 0, it follows that

$$2y = y' - 2 \Rightarrow 2y + 2 = \frac{dy}{dx} \Rightarrow \frac{dy}{y+1} = 2\,dx.$$

Integrating gives

$$\ln|y+1| = 2x + C \Rightarrow y = \pm e^C \cdot e^{2x} - 1.$$

Furthermore,

$$y(0) = \int_0^0 f(t)\,dt = 0,$$

so $\pm e^C \cdot e^0 - 1 = 0$. Thus $y(x) = e^{2x} - 1$ and $f(x) = 2e^{2x}$.

9. A free finitely generated group Γ of rank 2 is the group of all words generated by the two letters γ_1 and γ_2. Each γ not equal to the identity element e, can be uniquely written as $\gamma_1^{p_1}\gamma_2^{p_2}\gamma_1^{p_3}\cdots\gamma_{i_k}^{p_k}$, or as $\gamma_2^{p_1}\gamma_1^{p_2}\gamma_2^{p_3}\cdots\gamma_{i_k}^{p_k}$, where $p_1,\cdots,p_k$ are non-zero integers, and $i_k = 1,2$. The norm norm $||\gamma||$ is then defined to be $\sum_{i=1}^k |p_i|$. If Γ is Abelian the words can be simplified as $\gamma_1^p\gamma_2^q$, $p,q \in \mathbb{Z}$. *The ball of radius* r centered at e, $B(r)$, consists of all $\gamma \in \Gamma$ with $||\gamma|| \leq r$. We denote by $\#B(r)$ the number of elements in $B(r)$. Prove that:

   (a) If Γ is Abelian, $\#B(r) = 2N^2 + 2N + 1$, where $N = \lfloor r \rfloor$.

   (b) If Γ is non-Abelian, $\#B(r) = 2 \cdot 3^N - 1$, where $N = \lfloor r \rfloor$.

   Solution: Since $||\gamma||$ is an integer, without loss of generality, we may assume $r = N$ is an integer.

   (a) We need to find the number of solutions to the inequality $|p| + |q| \leq N, p, q \in \mathbb{Z}$. If $N = 0,1$ the solution is trivial, so assume $N > 1$. If $p = q = 0$, there is one solution. If $p = 0, q \neq 0$, we have $|q| \leq N$, so there are $2N$ solutions. Similarly, if $p \neq 0, q = 0$, there are also $2N$ solutions.

   If $p \neq 0, q \neq 0$, then the number of solutions for $|p|+|q| \leq n$ is 4 times the number of soltions to the inequality $p + q \leq N$ with $p, q > 0$, which is the same as the number of solutions to the equation $p + q + k = N$ with $p, q > 0$ and $k \geq 0$. This equation has

$$\sum_{k=2}^{N}(k-1) = \frac{N\cdot(N-1)}{2}$$

solutions. Thus $|p| + |q| \le N$ has $4 \cdot \frac{N(N-1)}{2} = 2N(N-1)$ solutions where neither p nor q are 0.

Finally, the number of solutions to $|p| + |q| \le N, p, q \in \mathbb{Z}$ is

$$2N^2 - 2N + 2N + 2N + 1 = 2N^2 + 2N + 1.$$

(b) We first prove the following lemma:

Lemma: The number of solutions to the equation

$$|x_1| + |x_2| + \cdots + |x_k| + |x_{k+1}| = r$$

where $x_i \ne 0$ for $i = 1, 2, \ldots, k$ and $x_{k+1} \ge 0$, is $2^k \binom{r}{k}$.

Proof of Lemma: The number of solutions to the above equation is 2^k times the number of solutions to $x_1 + \cdots + x_k + x_{k+1} = r$, with the same restrictions on the x's. Writing $x_i = x_i' + 1$, we get the equivalent equation

$$x_1' + x_2' + \cdots + x_k' + x_{k+1}' = r - k,$$

where $x_i' \ge 0, 1 = 1, \ldots, k$ and $x_{k+1} \ge 0$. This equation has

$$\binom{k+1+r-k-1}{r-k} = \binom{r}{k}$$

solutions. □

We count the number of nonzero elements γ starting with γ_1, that is, $\gamma = \gamma_1^{p_1}\gamma_2^{p_2}\cdots\gamma_{i_k}^{p_k}$ where $||\gamma|| \le N$. $||\gamma|| \le N$ implies that $|p_1| + |p_2| + \cdots + |p_k| \le N, p_i \ne 0, i = 1, \ldots, k$. By the lemma, there are $2^k\binom{N}{k}$ solutions. Thus, the number of nonzero elements γ starting with γ_1 with $|\gamma| \le N$ is

$$\sum_{k=1}^{N} 2^k \binom{N}{k} = \left[\sum_{k=0}^{N} \binom{N}{k} \cdot 2^k\right] - 1 = (1+2)^N - 1 = 3^N - 1.$$

Similarly, the number of nonzero elements γ starting with $\gamma_2, |\gamma| \le N$ is also $3^N - 1$. Thus,

$$\#B(\gamma) = \#\{\gamma \in \Gamma : ||\gamma|| \le N\} = (3^N - 1) + (3^N - 1) + 1 = 2 \cdot 3^N - 1.$$

10. The n-dimensional unit sphere S^n, is the set of all points in $\mathbb{R}^{n+1}$ of distance 1 from the origin, that is $S^n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\}$. The intersection of every n-dimensional vector space $V \subset \mathbb{R}^{n+1}$ with S^n is a $(n-1)$-dimensional unit sphere, called a *great sphere* of S^n. Every great sphere divides S^n into two hemispheres. A hemisphere together with its boundary, is a *closed* hemisphere. Prove that given any $n+3$ points in S^n, there is a closed hemisphere that contains $n+2$ of them.

Solution: Label the points $p_1, p_2, \ldots, p_{n+3}$. Let $V \subseteq \mathbb{R}^{n+1}$ be an n-dimensional vector space that includes points $p_1, \ldots, p_n$. Then V divides S^n into two hemispheres S_1^n, S_2^n. By the pigeonhole principle, one of the hemispheres, say S_1^n, must contain 2 of the 3 remaining points. Thus S_1^n together with its boundary contains $n+2$ points.

(If V includes 2 or 3 of $P_{n+1}, p_{n+2}, p_{n+3}$, then the proof is obvious. If V includes 1 of these 3 points, any of the hemispheres that includes at least a point can be considered.)